

# Branching annihilating random walks with parity conservation on a square lattice

György Szabó<sup>1,2</sup> and Maria Augusta Santos<sup>2</sup>

<sup>1</sup> *Research Institute for Technical Physics and Materials Science  
POB 49, H-1525 Budapest, Hungary*

<sup>2</sup> *Departamento de Física and Centro de Física do Porto  
Faculdade de Ciências, Universidade do Porto  
Rua do Campo Alegre 687, 4150 Porto, Portugal*

Using Monte Carlo simulations we have studied the transition from an “active” steady state to an absorbing “inactive” state for two versions of the branching annihilating random walks with parity conservation on a square lattice. In the first model the randomly walking particles annihilate when they meet and the branching process creates two additional particles; in the second case we distinguish particles and antiparticles created and annihilated in pairs. Quite distinct critical behavior is found in the two cases, raising the question of what determines universality in this kind of systems.

PACS numbers: 64.60.Ht, 05.40.Fb, 02.50.-r

Branching annihilating random walks (BARWs) have been extensively studied in recent years because they are a prototype for a variety of reaction-diffusion-like systems (for recent reviews see the work by Cardy and Täuber [1] and by Marro and Dickman [2] for more general aspects). For different models, the random walkers can represent either domain walls (kinks) or active sites on a lattice. In general, the corresponding critical behavior belongs to the directed percolation (DP) universality class. According to the “DP conjecture” [3], most of the one-component models with a single absorbing state belong to the DP universality class. Exceptions can appear when additional symmetries or conservation laws are introduced. The best known one-dimensional (1D) exceptions are models in which, either the parity of the number of particles is conserved during the elementary processes [4] - [9], or there is an equivalence between two absorbing states [10,11]. Henceforth, we will concentrate on this type of 2D BARWs whose behavior differs from the 2D DP universality class [12].

A field theoretical analysis of such systems was recently reported [13,1]. In 1D, the numerical and theoretical approaches are in satisfactory agreement: a new universality class appears when parity is preserved. Agreement is also found for  $D > 2$  in which case the mean-field results are valid. The nonexistence of the absorbing state (extinction) for finite values of the branching rate predicted by mean-field approximation for  $D \geq 2$  was also confirmed by early Monte Carlo (MC) simulations [14] in 2D and 3D. However, according to the field theoretical analysis, logarithmic corrections are expected for the marginal dimensionality  $D = 2$  for models with parity conservation and this was not born out by previous work. In this communication we report results of extensive MC simulations in 2D lattices showing evidence of such corrections. Inspired by the kink/antikink interpretation of creation and annihilation, we have also considered a modification of the model where two types of particles are present in

the creation and annihilation processes and indeed found that a distinct behavior appears in this case.

In the first model there is only one species of particles walking randomly on a square lattice. In addition to the single particle diffusion, the time evolution is governed by creation and annihilation of particle pairs as follows. A randomly chosen particle creates two additional particles with a probability  $p$  which are located on two randomly chosen neighboring empty sites [15]; otherwise the chosen particle jumps to one of its nearest neighbor positions. For both elementary processes, if the destination site is occupied then the two particles annihilate. In order to study extinction, we consider only even initial numbers of particles.

In the second model we distinguish particles and antiparticles, the creation and annihilation processes involving a particle/antiparticle pair with an evolution rule similar to the above one. During the sequential updating we neglect all the elementary processes which would result in two particles (or antiparticles) on the same site. The numbers of particles and antiparticles are chosen to be equal. As a result we have a unique absorbing state (no particles) independent of time. This model can be considered as a 2D generalization of the parity conserving model introduced by Menyhárd [5] where the 1D ferromagnetic domain walls are represented by particles and antiparticles. It is worth mentioning that in the former model the particles and antiparticles are alternately positioned along the chain and this feature is maintained by the elementary processes, therefore their distinction is not relevant. In the present 2D model, however, two particles (antiparticles) can occupy neighboring positions and they can avoid each other.

The MC simulations are performed on an  $L \times L$  square lattice with periodic boundary conditions for different values of the branching rate  $p$ . In order to have sufficiently accurate results, the system size ( $L$ ) is increased up to  $L = 2000$  for small concentration of particles. The

simulations are started from a randomly half-filled lattice and during the evolution we record the concentration of particles (other initial conditions were also tested). Time is measured in Monte Carlo steps (MCS) within which each particle has an opportunity to jump (probability  $1 - p$ ) or branch (probability  $p$ ).

For both models, in the absence of branching ( $p = 0$ ), the number of particles decreases monotonously and eventually vanishes. For  $p > 0$ , however, the system remains active with a fluctuating number of particles  $n$  if the lattice size is sufficiently large. In the stationary state the average concentration ( $c = \langle n \rangle / L^2$ ) of particles vanishes continuously when  $p$  tends to 0. In other words, decreasing the branching rate, both systems undergo a transition with a critical point  $p_c = 0$ .

First we have investigated the decrease of concentration at the critical point ( $p_c = 0$ ). For this purpose the time-dependent average concentration and its fluctuation  $\chi = L^2 \langle (c - n/L^2)^2 \rangle$  are determined at discrete time steps (equidistant in the logarithmic scale) by averaging over 500 runs.

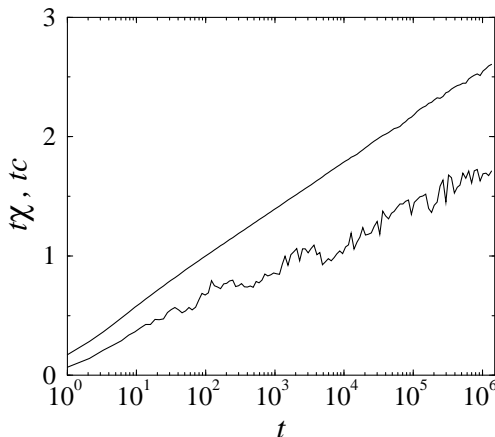


FIG. 1. Concentration (upper curve) and its fluctuation (lower curve) multiplied by the time *vs.* the logarithm of time at zero branching rate for the first model.

For the first model the field theoretical investigation suggests  $c \propto \ln t / t$  [1]. To check this prediction we have plotted  $tc$  *vs.*  $t$  in a log-lin plot. Our MC results (see Fig. 1) obtained for  $L = 2000$  indicate clearly that the time-dependent concentration can be well described as

$$c(t) = \frac{A + B \ln t}{t} \quad (1)$$

for sufficiently long times ( $t > 100$  MCS). This function fits the MC data if  $A = 0.2238$  and  $B = 0.8979$ . The leading term of this asymptotic behavior agrees with the above prediction given by Lee [16] and Cardy and Täuber [1].

The fluctuation of concentration decreases proportionally with the average value of concentration as demonstrated in Fig. 1. Neglecting the “noisy decoration” due

to the statistical error, the ratio of the time-dependent fluctuation and concentration can be well approximated as  $\chi(t)/c(t) \simeq 0.67(2)$  on the time range indicated in Fig. 1. This ratio agrees very well with the theoretical prediction ( $2/3$ ) obtained by Lee [16] using renormalization group technique.

For the second model the decrease of concentration follows a different behavior at the critical point. Indeed when  $p = 0$  we get the diffusion limited surface reaction  $A+B \rightarrow \emptyset$  case, already studied by several authors [17–19]. The concentrations decrease as  $t^{-d/4}$  if the dimension  $d$  is lower than 4 (the upper critical dimension for this system). Compared to the former case, the results depend more strikingly on size, as shown in a log-log plot (Fig. 2). The sharp decrease in  $c$  is a consequence of the extinction whose probability is higher for smaller systems. Our simulation confirms the mentioned power law behavior in the limit  $L \rightarrow \infty$  as indicated in Fig. 2. The extent of this behavior for the largest system is illustrated in the figure by the dashed line. The fluctuations are almost constant during this scaling regime ( $\chi(t) \simeq 0.009(1)$  for  $L = 2000$  and  $600 < t < 60000$ ).

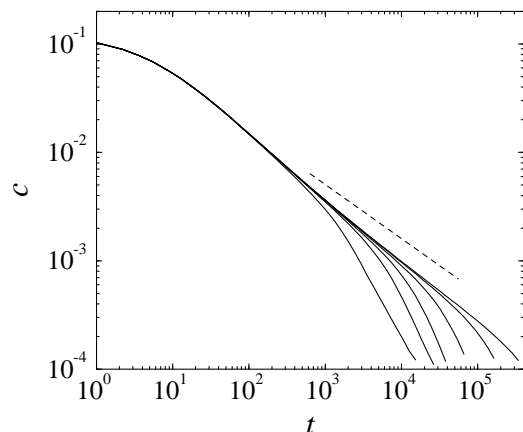


FIG. 2. Time dependence of concentration of particles (and antiparticles) in the second model at zero branching rate for different system sizes ( $L = 100, 200, 300, 500, 1000$  and  $2000$  from left to right). The dashed line (slope  $-0.5$ ) indicates the theoretical power law behavior.

Systematic and extensive MC simulations have been performed to study the average concentration and its fluctuation in the steady state (reached after some thermalization) for finite branching rates. For the smallest  $p$  values both the thermalization and sampling times were longer than  $10^5$  ( $10^6$ ) MCS for the first (second) model. These simulations were repeated 20 times to suppress the undesired effects of long time fluctuations.

The results for the first model are summarized in Fig. 3. At first glance, the MC data for the concentration (diamonds in the log-log plot) indicates a power law behavior, namely  $c(p) \propto p^\beta$  with  $\beta = 1.276$ , similar to what was found by Takayasu and Tretyakov [14]. The

careful reader can, however, observe a definite deviation from this behavior (positive curvature) whose magnitude exceeds our statistical error.

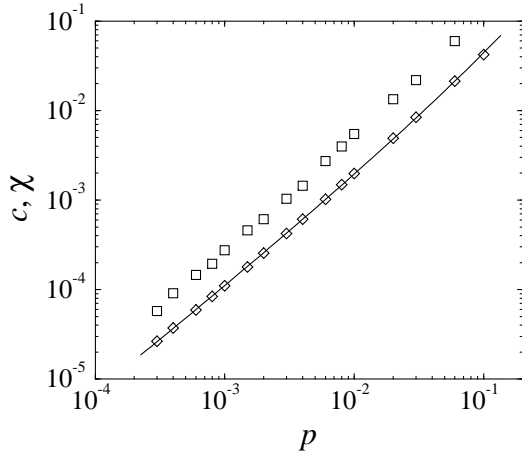


FIG. 3. Log-log plot of the particle concentration (diamonds) and its fluctuation (squares) as a function of branching rate for the first model. The solid line indicates the fitted curve described in the text.

Taking the logarithmic corrections into account Cardy and Täuber have suggested [13] that the leading term of the  $c(p)$  function is proportional to  $p/\ln^2(p)$ . This function does not fit adequately the present MC data; however, an excellent fitting is found if we use  $c(p) = p/[A + B \ln(p) + C \ln^2(p)]$ . This function is represented by a solid line in Fig. 3 for  $A = 0.4098$ ,  $B = -0.5825$  and  $C = 0.09601$ . Notice that this formula confirms the theoretical prediction in the limit  $p \rightarrow 0$ .

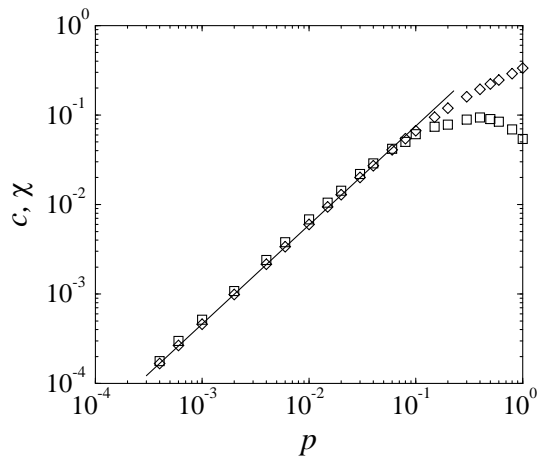


FIG. 4. Particle and antiparticle concentration (diamonds) and its fluctuation (squares) vs. branching rate for the second model. The fitted asymptotic power law behavior is indicated by the straight line (slope 1.11).

Figure 3 indicates clearly that the concentration fluctuation is proportional to the concentration itself in the

$p$  region we have studied. These quantities satisfy the relation  $\chi(p)/c(p) = 2.5(2)$  within the statistical error.

Similar investigations have been performed for the second model. Figure 4 demonstrates clearly that the MC data tends towards a power law for both the concentration ( $c$ ) and its fluctuation ( $\chi$ ) at small values of branching  $p$ . Fitting the function  $c(p) = Ap^\beta$  to our numerical data we have obtained that  $\beta = 1.11(1)$ . The ratio of the fluctuation to the concentration is smaller than found for the first model, namely,  $\chi(p)/c(p) = 1.10(5)$ .

In summary, we have studied and compared two simple models of BARWs on a square lattice with parity conservation. In contrast with the first model – where there is only one type of particles – the second model has particles and antiparticles annihilating (only) each other when they meet. This distinction has caused significant differences between their behavior at the critical point (no branching) as well as in the stationary states for finite branching rate. For the first model, our MC simulations have justified the appearance of the logarithmic corrections predicted theoretically by Lee and by Cardy and Täuber. On the contrary, we have observed power law behavior in the second model. Surprisingly, the fluctuations decrease with  $p$  – more precisely, the  $p$  dependence of the fluctuation is found to be proportional to the concentration for both models. The significant differences between the behavior of the present models imply the possibility to find other two-dimensional systems whose critical behavior does not belong to the directed percolation universality class. The belief that in these systems parity conservation is sufficient to determine the universality class is probably also to be questioned.

## ACKNOWLEDGMENTS

We thank U. Täuber for his valuable comments. We have benefited from discussions with J. F. Mendes and M. C. Marques. G. S. acknowledges a senior research fellowship from PRAXIS (Portugal). Supports from NATO (CRG-970332), PRAXIS (project PRAXIS/2/2.1/Fis/299/94) and the Hungarian National Research Fund (T-16734) are also acknowledged.

- 
- [1] J. L. Cardy and U. C. Täuber, J. Stat. Phys. **90**, 1 (1998).
  - [2] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, 1998).
  - [3] H. K. Janssen, Z. Phys. B **42**, 151 (1981); P. Grassberger, Z. Phys. **47**, 365 (1982).
  - [4] P. Grassberger, F. Krause and T. von der Twer, J. Phys. A **17**, L105 (1984); P. Grassberger, J. Phys. A **22**, L1103 (1989).

- [5] N. Menyhárd, J. Phys. A **27**, 6139 (1994); N. Menyhárd and G. Ódor, J. Phys. A **28**, 4505 (1995).
- [6] D. ben-Avraham, F. Leyvraz, and S. Redner, Phys. Rev. E **50**, 1843 (1994).
- [7] I. Jensen, J. Phys. A **26**, 3921 (1993); Phys. Rev. E **50**, 3623 (1994).
- [8] N. Inui and A. Yu. Tretyakov, Phys. Rev. Lett. **80**, 5148 (1998).
- [9] M. C. Marques and J. F. F. Mendes, preprint (cond-mat/9811015).
- [10] M. H. Kim and H. Park, Phys. Rev. Lett. **73**, 2579 (1994).
- [11] H. Hinrichsen, Phys. Rev. E **55**, 219 (1997).
- [12] R. C. Brower, M. A. Furman, and M. Moshe, Phys. Lett. **76B**, 213 (1978); I. Jensen, H. C. Fogedby, and R. Dickman, Phys. Rev. A **41**, 3411 (1990).
- [13] J. L. Cardy and U. C. Täuber, Phys. Rev. Lett. **77**, 4780 (1996).
- [14] H. Takayasu and A. Yu. Tretyakov, Phys. Rev. Lett. **68**, 3060 (1992).
- [15] In references [14] and [7], the branching probability is denoted by  $1 - p$ .
- [16] B. P. Lee, J. Phys. A: Math. Gen. **27**, 2633 (1994).
- [17] D. Toussaint and F. Wilczek, J. Chem. Phys. **78**, 2742 (1983).
- [18] S. Cornell, M. Droz, and B. Chopard, Physica A **188**, 322 (1992).
- [19] B. P. Lee and J. Cardy, J. Stat. Phys. **80**, 971 (1995).